Closing Today: 15.4
Closing Thu: 15.5

Midterm 2 is Tuesday, March 1

Quick list of topics:

13.3/4: Curvature, Arc Length, TNB-Frame,
Normal Plane, Osculating Plane
Position, Velocity, Acceleration
Tangent and Normal Comp. of Accel.

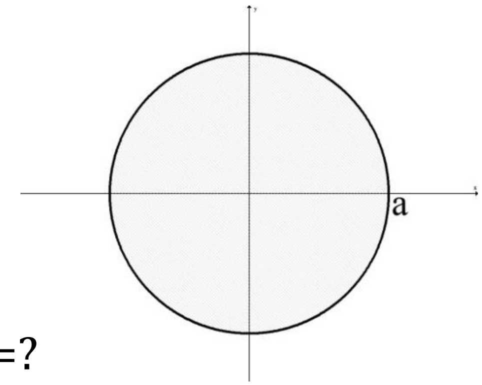
14.1/3/4/7: Level Curves, Domain, Partials,
Tangent Plane (Linear approx/Differential)
Local max/min (2nd deriv. test)
Global max/min

15.1-15.4: Definition of double integral,
volume, area and average value apps,
general regions (top/bot, left/right),
reversing order, polar

Entry Task: Give the bounds for the region in polar coordinates and set up the two double integrals:

1. 'a' is a constant,

$$\iint_D 1 dA = ?$$



$$\iint_D \sqrt{a^2 - x^2 - y^2} dA = ?$$

15.5 Center of Mass

We have seen:

$$\iint_R f(x,y)\,dA = \text{Volume under f(x, y) over } R$$

$$\iint_R 1\ dA = \text{Area of R}$$

$$\frac{1}{Area}\iint_R f(x,y)\,dA = \text{Average of f(x, y) over } R$$

Today we will see one more application, which is a generalization of an application from Math 125.

Goal: Given a thin uniformly distributed plate (a *lamina*) with density at each point p(x,y) can we find the center of mass (*centroid*).

p(x,y) = mass/area (kg/m^2)

Motivation:

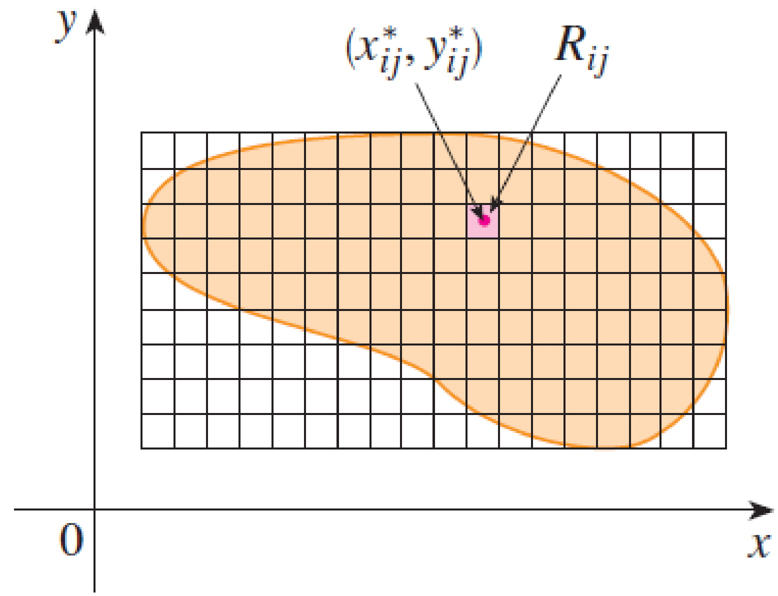
In general: If you are given ***n* points**

$(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ with corresponding masses $m_1, m_2, \ldots, m_n$

then

$$\bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1y_1 + \cdots + m_ny_n}{m_1 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_iy_i}{\sum_{i=1}^{n} m_i} = \frac{M_x}{M}$$



Now consider a thin plate with density at each point given by p(x,y). Here is the derivation of the center of mass formulas:

1. Break region into m rows and n columns.
2. Find the center of mass of each rectangle.
$$(\bar{x}_{ij}, \bar{y}_{ij})$$
3. Estimate the mass of each rectangle.
$$m_i = p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A$$
4. Now use the formula for *n* points.
5. Take the limit.

Center of Mass:

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R \ xp(x,y)dA}{\iint_R \ p(x,y)dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R \ yp(x,y)dA}{\iint_R \ p(x,y)dA}$$

Example:

Consider a 1 m by 1 m square metal plate. The density is given by $p(x,y) = kx$ kg/m^2 for some constant k.

> Side note: This means that the density is proportional to the distance from the y-axis (in other words it gets heavier, at a constant rate, from left-to-right).

Find the center of mass.

Note:

Proportional to the distance from the y-axis

$$p(x, y) = kx.$$

Proportional to the distance from the x-axis

$$p(x, y) = ky.$$

Proportional to the distance from the origin:

$$p(x, y) = k\sqrt{x^2 + y^2}.$$

Proportional to the square of the distance from the origin: $p(x, y) = k(x^2 + y^2)$.

Inversely proportional to the distance from the origin: $p(x, y) = \frac{k}{\sqrt{x^2+y^2}}$.

Example:

A thin plate is in the shape of the region bounded between the circles of radius 1 and 2 in the first quadrant. The density is proportional to the distance from the origin. Find the center of mass.